**Complex Differentiation**

**Derivatives:** If  is single-valued in some region  of the  plane, then the derivative of is defined as



provided that the limit exists independent of the manner in which .

Alternatively, if is defined in some neighbourhood of , then the derivative of at  is defined as



provided that the limit exists independent of the manner in which .

**Analytic (regular or holomorphic) function:** A complex function  is said to be analytic at a point , if its derivative exists not only at but also at each point in some neighbourhood of .

**Cauchy-Riemann equations:** A necessary condition is that if  be analytic in a region , then  and  satisfy the Cauchy-Riemann equations

 and  (1)

If the partial derivatives of (1) are continuous in , then the Cauchy-Riemann equations are sufficient conditions that  be analytic in .

**Harmonic Function:** Function such as  which satisfies the Laplace’s equation  in a region is called harmonic function and is said to be harmonic in .

**Harmonic conjugate:** The function  is said to be a harmonic conjugate of  if  and  are harmonic and satisfy Cauchy-Riemann equations.

**L’Hospital’s rule:** If and  be analytic in a region containing the point  and suppose that  but . Then L’Hospital’s rule states that

.

In case , the rule may be extended.

**Singular point:** A point at which  fails to be analytic is called a singular point or singularity of . Various types of singularities exist such as:

1. **Isolated singularity:** The point  is called an isolated singular point of if we can find  such that the circle  encloses no singular point other than . That is, there exists a deleted neighbourhood  in which  is analytic. There are three types isolated singular points such as pole, removable singular point and essential singular point.

Example:  has an isolated singularity at .

1. **Pole:** If we can find a positive integer  such that , then  is called a pole of order . If ,  is called a simple pole.

Example: (a).  has a pole of order  at .

(b).  has a pole of order  at , and simple poles at  and .

**3. Branch point:** The branch point of multiple-valued function have already been studied which

are also singular points.

Example: (a).  has a branch point at .

(b).  has branch points where , i.e. at and .

**4. Removable singularity:** The singular point  is called a removable singularity of  if

 exists.

Example: (a). The singular point  is a removable singularity of  since .

**5. Essential singularity:** A singularity which is not a pole, branch point or removable singularity is

called an essential singularity.

Example: (a).  has an essential singularity at .

**6. Singularity at Infinity:** The function  has a singularity at  if is a singularity of

 .

Example: The function  has a pole of order at , since  has a pole of order  at .

**Complex differential operators:** Let  is a real continuously differentiable function of  and while  is a complex continuously differentiable function of  and .

Since  and . So in terms of conjugate coordinates, we have



and 

Since  is any continuously differentiable function so





and 



Now 

and 

Here, and  are complex differential operators.

**Gradient, Divergence, Curl and Laplacian:** Let  is a real continuously differentiable function of  and while  is a complex continuously differentiable function of  and . Since  and . So in terms of conjugate coordinates, we have



and 

1. **Gradient:** We define the gradient of a real function (scalar) by



Geometrically, this represents a vector normal to the curve  where  is a constant.

Similarly, the gradient of a complex function (vector) is defined by





In particular if  is an analytic function of then  and so the gradient is zero, i.e.  and

, which shows that the Cauchy-Riemann equations are satisfied in this case.

1. **Divergence:** We define the divergence of a complex function (vector) by









Similarly we can define the divergence of a real function. It should be noted that the divergence of a complex or real function (vector or scalar) is always a real function (scalar).

1. **Curl:** We define the curl of a complex function (vector) by









Similarly we can define the curl of a real function.

1. **Laplacian:** The Laplacian operator is defined as the dot or scalar product of  with itself.

i.e.







Note that if is analytic,  so that  and , i.e.  and  are harmonic.

**Problems**

**Problem-01:** Show that does not exist anywhere.

**Solution:** Here 

By definition we have



if this limit exists independent of the manner in which  approaches zero.

Now 











Taking limit along real axis , we get



Again, taking limit along imaginary axis , we get



The above two limits are not equal, that is, the limit depends on manner in which .

Hence  does not exist anywhere. **(Showed)**

**Problem-02:** Show that does not exist anywhere.

**Solution:** Here 

By definition we have



if this limit exists and independent of the manner in which  approaches zero.

Now 















In different way  gives different values, so that  does not exist anywhere. **(Proved)**

**Problem-03:** State and prove Cauchy-Riemann Equations.

OR

State and prove necessary condition for a function to be analytic.

OR

State and prove sufficient condition for a function to be analytic.

OR

Prove(a)necessary and (b) sufficient condition that  is analytic, iff it satisfy

the Cauchy-Riemann equations , .

**Solution:** **Sufficient condition:** Let,  be a function defined in a region . If in , the Cauchy Riemann equations are satisfied and ,  are continuous then  is analytic in .

**Proof:** Since  and  are continuous in .

Then we have







where  as  and  as .

Again  and  are continuous in .

Then we have







where  as  and  as .

Now, 





where  as  and  as .



By using Cauchy-Riemann equations, we get







Then on dividing by  and taking the limit as , we see that







Since the derivative exists. Hence  is analytic. **(Proved)**

**Necessary condition: :** Let,  be a function defined in a region . The necessary condition for to be analytic in is that the Cauchy Riemann equations  and  are satisfied in .

**Proof:** Let  be analytic in .

Then at any point ,





must exist independent of the manner in which  ( or  and ) approaches zero.

Taking limit along real axis (, ), we get







Again, taking limit along imaginary axis (, ), we get







Since  is analytic, then two limits (1) and (2) must be equal.

Hence, 

Now, equating real and imaginary part on both sides, we have

 and 

which are the Cauchy-Riemann equations. **(Proved)**

**Problem-04:** Prove that in polar form the Cauchy-Riemann equations can be written as

 and 

**Solution:** By relation of Cartesian coordinate  and Polar coordinate  , we have

 and 

From (1) and (2), we get

and 

x

***r***

**O**

**X’**

**X**

**Y**

**Y’**

**P(r,θ)**

**θ**

**P(x,y)**

y

Differentiating (3) and (4) with respect to *x* and *y*, we get









Now 







Again, 







By Cauchy- Riemann equations, we get







Again, 





Multiplying (5) by and (6) by  and adding, we get







Again, multiplying (5) by and (6) by  and subtracting, we get







Hence, the Cauchy- Riemann equations in polar form are

 and  **(Proved)**

**Problem-05:** Prove that is nowhere analytic

**Solution:** Given that 

Let 





Here  and 

Now , ,  and 

The above equations show that , ,  and  are continuous everywhere. But the Cauchy-Riemann equations are satisfied only at the origin. Hence  is only the point at which  exists. Thus 

is nowhere analytic. **(Proved)**

**Problem-06:** Prove that an analytic function with constant modulus is a constant.

**Solution:** Let  be an analytic function.

 where  is a constant





Differentiating with respect to *x*, we get





Similarly,

Using Cauchy-Riemann equations in (2), we get



Squaring (1) and (3) and then adding, we have







The equation (4) will be valid if



which implies

 

Integrating, we get

 where  is a constant

Hence  is a constant function. **(Proved)**

**Problem-07: If** and  are harmonic in a region , then prove that is analytic.

OR

**If** and  satisfy Laplace’s equation, then show  is analytic.

where  and .

**Solution:** Since  and  are harmonic function in a region *R.*



and



Let 

Here 

and 

Differentiating (3) and (4) with respect to *x* and *y* respectively, we get









Subtracting (8) from (5), we get









Adding (6) and (7), we get









Since, the Cauchy-Riemann equations are satisfied so  is analytic in *R*. **(Proved)**

**Problem-08: If** is an analytic function of , then prove that .

**Solution:** Let  be an analytic function*.*

Then 

and 





Now, 



Adding (1) and (2), we get





 Since is harmonic 



 **(Proved)**

**Problem-09:** Prove that the real and imaginary parts of an analytic function of a complex variable when expressed in polar form satisfy the equation .

**Solution:** ‍We know that the Cauchy-Riemann equations in polar form are





Differentiating (1) with respect to , we get



Differentiating (2) with respect to , we get



We know that







Similarly, 

 **(Proved)**

**Problem-10:** Show that is differentiable at .

**Solution:** Here 

By definition we have



if this limit exists independent of the manner in which  approaches zero.

Now 









Taking limit along real axis , we get



Again, taking limit along imaginary axis , we get



The above two limits are equal, that is, the limit does not depend on manner in which .

Hence  is differentiable at . (**Showed**)

**Problem-11:** If is analytic in a regionand if  and  have continuous second order partial derivatives in , then show that  and  are harmonic in .

**Solution:** Given  is analytic in the region . By Cauchy-Riemann equations we have

 (1)

 (2)

Again given  and  have continuous second order partial derivatives in . So we have

 (3)

 (4)

Now from (3) we get









Thus,  satisfies Laplace equation and hence it is harmonic.

Again, from (4) we get









Thus,  satisfies Laplace equation and hence it is harmonic. (**Showed**)

**Problem-12:** Prove that, ifa function  is differentiable at a point, then is continuous at that point, but the converse is not necessarily true.

**Solution:** Let the function  is differentiable at .

Now 









Hence  is continuous at . Thus every differentiable function is continuous.

**Converse part:** The converse of the given statement is not true. We shall prove this by the following counter example.

Let 

Now 





Taking the limit along real axis , we have



Taking the limit along imaginary axis , we have



Since the above two limit are equal so  exists and equal to the functional value at ,

i.e. 

Hence  is continuous at .

Again at , we have 





Taking limit along real axis , we get



Taking limit along imaginary axis , we get



The above two limits are not equal, that is, the limit depends on manner in which .

Hence  is not differentiable at  . **(Showed)**

**Problem-13:** Prove that the function  is harmonic. Find its harmonic conjugate  and express  as an analytic function of .

**Solution:** Given that  (1)

**1st part:** Differentiating (1) with respect to , we get

 (2)

 (3)

Again, differentiating (1) with respect to , we get

 (4)

 (5)

Adding (3) and (5), we get



.

Since  satisfies the Laplace’s equation so it is harmonic. **(Proved)**

**2nd part:** If is harmonic conjugate of , then by Cauchy-Riemann equations, we have



 (6)

Integrating (6) with respect to , we get

 (7)

Differentiating (7) with respect to , we get







 (8)

Integrating (8) with respect to , we get



From (7), we have



This is the required harmonic conjugate of . **(Ans)**

**3rd part:** Let 







 where 

 **(Ans)**

**Problem-14:** Prove that the function  is harmonic. Find its harmonic conjugate.

**Solution:** Given that  (1)

**1st part:** Differentiating (1) with respect to , we get

 (2)

 (3)

Again, differentiating (1) with respect to , we get

 (4)

 (5)

Adding (3) and (5), we get



.

Since  satisfies the Laplace’s equation so it is harmonic. **(Proved)**

**2nd part:** If is harmonic conjugate of , then by Cauchy-Riemann equations, we have



 (6)

Integrating (6) with respect to , we get

 (7)

Differentiating (7) with respect to , we get







 (8)

Integrating (8) with respect to , we get



From (7), we have



This is the required harmonic conjugate of . **(Ans)**

**Problem-15:** Show that  is harmonic. Find  such that  is analytic.

**Solution:** Given that  (1)

**1st part:** Differentiating (1) with respect to , we get

 (2)



 (3)

Again, differentiating (1) with respect to , we get

 (4)



 (5)

Adding (3) and (5), we get



.

Since  satisfies the Laplace’s equation so it is harmonic. **(Showed)**

**2nd part:** If is harmonic conjugate of , then by Cauchy-Riemann equations, we have



 (6)

Integrating (6) with respect to , we get





 (7)

Differentiating (7) with respect to , we get







 (8)

Integrating (8) with respect to , we get



From (7), we have



This is the required harmonic conjugate of . **(Ans)**

**3rd part:** Let 

















 where 

 **(Ans)**

**Problem-16:** Prove that the function  is not analytic at origin but the Cauchy-Riemann equations are satisfied.

**Solution:** Given that 



Here, 

and 

Now  and 

At , we get









and 







Similarly, at , we get









and 







From the above relations, we have

 and 

Hence, the Cauchy-Riemann equations are satisfied at origin.

Consider 





Taking the limit along real axis , we have





Taking the limit along imaginary axis , we have





Taking the limit along the path, we have







which is different from the above limits.

Therefore  does not exist and so  is not analytic at origin. **(Proved)**

**Problem-17:** Prove that the function  is not analytic at origin but the Cauchy-Riemann equations are satisfied.

**Solution:** Given that 



Here, 

and 

Now  and 

At , we get







and 





Similarly, at , we get







and 





From the above relations, we have

 and 

Hence, the Cauchy-Riemann equations are satisfied at origin.

Consider 





Taking the limit along real axis , we have





Taking the limit along imaginary axis , we have





Taking the limit along the path, we have





which is different from the above limits.

Therefore  does not exist and so  is not analytic at origin. **(Proved)**

**Problem-18:** Prove that the function  is not analytic at origin but the Cauchy-Riemann equations are satisfied.

**Solution:** Given that 



Here, 

and 

Now  and 

At , we get







and 





Similarly, at , we get







and 





From the above relations, we have

 and 

Hence, the Cauchy-Riemann equations are satisfied at origin.

Consider 





Taking the limit along real axis , we have





Taking the limit along imaginary axis , we have





Taking the limit along the path, we have





which is different from the above limits.

Therefore  does not exist and so  is not analytic at origin. **(Proved)**

**Problem-19:** If  , find (a)  and (b) determine where is non-analytic.

**Solution:** We have 

1. 





1. The function  is analytic for all finite values of except  where the derivative does not exist and the function is non-analytic. The point is a singular point of .

**Problem-20:** For the function , locate and name all the singularities in the finite and also determine where  is analytic.

**Solution:** Given that 

In the finite  the singularities will be obtained by solving the equation







In the finite z-plane, the singular point  is a pole of order and  is a pole of order 2.

In the finite   is analytic everywhere except the points  and .

**Problem-21:** Determine the singular points of  in the finite z-plane

**Solution:** Given that 

The singular points are obtained by solving the equation







In the finite z-plane, the singular point and  are simple poles.

**Problem-22:** For the function , locate and name all the singularities.

**Solution:** Given that 

In the finite  the singularities will be obtained by solving the equation







In the finite z-plane, the singular point  is a pole of order 2.

To determine whether there is a singularity at  (the point at infinity), let .

Then 



Since  is a simple pole for the function  so  is a simple pole at infinity for the function .

**Exercise**

**Problem-01:** Prove that the function  is harmonic. Find its harmonic conjugate.

**Problem-02:** Show that  satisfies the Laplace’s equation. Find its harmonic conjugate 

such that  is analytic.

**Problem-03:** Prove that the function  is harmonic. Find its harmonic conjugate

 and express  as an analytic function of .

**Problem-04:** Prove that the function  is harmonic. Find its harmonic conjugate

 and express  as an analytic function of .

**Problem-05:** If  and , then show that both  and  satisfy the Laplace’s equation but  is not an analytic function.

**Problem-06:** For each of the following functions locate and name the singularities in the finite z-plane:

(a), (b) , (c) , (d) 

(e)  , (f) 

**Ans: (a)**  ; simple pole, (b) ; branch point, ; pole of order 2, **(c)** ; essential singularity. (d) , ; branch points. (e) ; pole of order,(f) ; branch point, ; pole of order 4.

**Problem-07:** Determine which of the following functions are harmonic. For each harmonic function find the conjugate harmonic function  and express  as an analytic function of .

1. 
2. 
3. 
4. 

**Ans:** (a) 

(b). Not harmonic

(c) 

(d) 

**Problem-08:** Verify that the Cauchy-Riemann equations are satisfied for the following functions:

1. 
2. ; 
3. 
4. 
5. 
6. 
7. 
8. 
9. 

N.T: For solution see the book Complex analysis- A.K.M. Shahidullah